

# The $su(1, 1)$ Dynamical Algebra for the Generalized MICZ-Kepler Problem from the Schrödinger Factorization

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**Abstract** We apply the Schrödinger factorization to construct the generators of the dynamical algebra  $su(1, 1)$  for the radial equation of the generalized MICZ-Kepler problem.

**Keywords** MICZ-Kepler · Lie algebras · Dynamical algebras

## 1 Introduction

The factorization methods have played an important role in the study of quantum systems [7, 13, 14, 26–28]. This is because, if the Schrödinger equation is factorizable, the energy spectrum and the eigenfunctions are obtained algebraically. Infeld and Hull [13, 14] used the ideas of Dirac [7] and Schrödinger [26–28] and created a factorization method (IHFM) which uses a particular solution for the Riccati equation. This method allowed to classify the problems according to the characteristics involved in the potentials and it is closely related to the supersymmetric quantum mechanics [2, 4, 8, 15].

The Schrödinger factorization [26] is a technique which is essentially different to that of the IHFM [13]. In order to clarify these differences we consider a typical central-potential problem like the two-dimensional hydrogen atom or the two-dimensional isotropic harmonic oscillator. The ladder operators obtained by means of the IHFM can only generate a finite

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number of states with different values of the angular momentum for each degenerate energy eigenvalue [5]. Therefore, the symmetry algebras of these problems for bound states are compact. On the other hand, the raising and lowering operators obtained by applying the Schrödinger factorization generate an infinite number of states with different values of the energy for each degenerate angular momentum eigenvalue. Thus, dynamical symmetry algebras for these systems are non-compact and have infinite-dimensional representations [5]. Historically the Schrödinger factorization has been less applied to physical problems than the IHFM.

A systematic method to find both compact and non-compact algebra generators for a given system has not still developed. These generators have been intuitively found and forced them to close an algebra, as it is extensively shown in [3, 9, 10, 16, 21, 29, 30]. For several one-dimensional potentials it has been shown they exhibit an  $su(1, 1)$  dynamical algebra [2, 10, 16, 21, 29]. It must be emphasize that these authors had to introduce an additional variable to define the generators of this algebra. On the other hand, in [19, 20] we have shown that by means of the Schrödinger factorization it is possible to construct the generators of the dynamical algebras for some central-potential problems without using any additional variable.

The MICZ-Kepler problem is the Kepler problem when the nucleus of this hypothetical hydrogen atom also carries a magnetic charge. This problem was independently discovered by McIntosh and Cisneros [22] and Zwanziger [31]. It has been shown that the MICZ-Kepler problem possesses a Runge-Lenz-type vector as constant of motion and the group  $O(4)$  as symmetry group [22]. These facts reflect a great similarity with the Coulomb problem. Different generalizations for the MICZ-Kepler problem have been studied. For example, Meng [23] has studied the MICZ-Kepler problem in all dimensions and Mardoyan [17, 18] has solved the Schrödinger equation for the generalized MICZ-Kepler Hamiltonian ( $\hbar = m = c = 1$ )

$$H = \frac{1}{2}(-i\nabla - s\mathbf{A})^2 + \frac{s^2}{2r^2} - \frac{1}{r} + \frac{c_1}{r(r+z)} + \frac{c_2}{r(r-z)}, \quad (1)$$

where  $\mathbf{A}$  is the magnetic vector potential of a Dirac monopole,

$$\mathbf{A} = \frac{1}{r(r-z)}(y, -x, 0), \quad (2)$$

such that  $\nabla \times \mathbf{A} = \frac{\mathbf{r}}{r}$ ,  $r$  is the distance from the electron to the hydrogen nucleus,  $s$  is the magnetic charge of the Dirac monopole which takes the values  $s = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \dots$  and,  $c_1$  and  $c_2$  are non-negative constants. Moreover, there are generalizations of the MICZ-Kepler systems on the three-dimensional sphere [12] and on the hyperboloid [24]. Recently, Giri studied the radial Schrödinger equation corresponding to the generalized MICZ-Kepler problem (1) from supersymmetric quantum mechanics [11]. In this way he obtained the superpotential from which the SUSY factorization operators are defined. It is immediate to show that the action of these operators on the radial eigenstates  $R_{nj}(r)$  of the Hamiltonian (1) is to change the secondary quantum number  $j$  leaving the principal quantum number  $n$  fixed.

In this work we apply the Schrödinger factorization [19, 26] to the MICZ-Kepler problem and obtain ladder operators which change the energy quantum number leaving fixed the total angular momentum quantum number. Because of what we emphasized above, these operators must be related to the algebras of the non-compact symmetry groups. In fact, we show how the Schrödinger operators allow us to construct the generators of the  $su(1, 1)$

dynamical algebra for the bound states of the radial equation for the generalized MICZ-Kepler Hamiltonian (1). The paper is organized as follows. In Sect. 2, we summarize the important points of the Mardoyan papers [17, 18] which are relevant to this work. In Sect. 3, the Schrödinger factorization is performed and the  $su(1, 1)$  dynamical algebra is constructed. Finally, in Sect. 4, we give the concluding remarks.

## 2 The Generalized MICZ-Kepler Problem in Spherical Basis

The Schrödinger equation  $H\Psi = E\Psi$  for the Hamiltonian (1) with  $\Psi \equiv R(r)Z(\theta, \phi)$  in spherical coordinates  $(r, \theta, \phi)$ , can be reduced to the uncoupled differential equations [11, 17, 18]

$$\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left( \sin\theta \frac{\partial Z}{\partial\theta} \right) + \frac{1}{4\cos^2\frac{\theta}{2}} \left( \frac{\partial^2}{\partial\phi^2} - 4c_1 \right) Z + \frac{1}{4\sin^2\frac{\theta}{2}} \left[ \left( \frac{\partial}{\partial\phi} + 2is \right)^2 - 4c_2 \right] Z = -AZ, \tag{3}$$

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) - \frac{\mathcal{A}}{r^2} R + 2 \left( E + \frac{1}{r} \right) R = 0, \tag{4}$$

where the quantized separation constant is given by

$$\mathcal{A} = \left( j + \frac{\delta_1 + \delta_2}{2} \right) \left( j + \frac{\delta_1 + \delta_2}{2} + 1 \right). \tag{5}$$

The unnormalized square-integrable solutions for (3) are [17, 18]

$$Z_{jm}^{(s)}(\theta, \phi; \delta_1\delta_2) = \left( \cos \frac{\theta}{2} \right)^{m_1} \left( \sin \frac{\theta}{2} \right)^{m_2} P_{j-m_+}^{(m_2, m_1)}(\cos\theta) e^{i(m-s)\phi}, \tag{6}$$

where  $m_1 = |m - s| + \delta_1 = \sqrt{(m - s)^2 + 4c_1}$ ,  $m_2 = |m + s| + \delta_2 = \sqrt{(m + s)^2 + 4c_2}$ ,  $m_+ = (|m + s| + |m - s|)/2$  and  $P_n^{(a,b)}$  are the Jacobi polynomials. The  $z$ -component of the total angular momentum  $m$  and the total angular momentum  $j$  take the quantized eigenvalues

$$m = -j, -j + 1, \dots, j - 1, j, \tag{7}$$

$$j = \frac{|m + s| + |m - s|}{2}, \frac{|m + s| + |m - s|}{2} + 1, \dots \tag{8}$$

Notice that the Dirac quantization condition  $s$  determines  $j$  and  $m$ . The equations above imply that  $j$  and  $m$  take integer or half-integer values depending on whether  $s$  takes integer or half-integer values.

By substituting the quantized separation constant (5) into (4) we obtained

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) - \frac{1}{r^2} \left( j + \frac{\delta_1 + \delta_2}{2} \right) \left( j + \frac{\delta_1 + \delta_2}{2} + 1 \right) R + 2 \left( E + \frac{1}{r} \right) R = 0. \tag{9}$$

This equation under the change  $j + (\delta_1 + \delta_2)/2 \rightarrow J$  results to be equal to the radial equation for the hydrogen atom [6]. Thus, the solutions for discrete spectrum of (9) apart from

normalization are [17, 18]

$$R_{nj}^{(s)} = (2\epsilon r)^{j + \frac{\delta_1 + \delta_2}{2}} e^{-\epsilon r} F(-n + j + 1, 2j + \delta_1 + \delta_2 + 2; 2\epsilon r), \tag{10}$$

where  $n = |s| + 1, |s| + 2, \dots$ ,  $F(a, b; z)$  is the confluent hypergeometric function, and the parameter  $\epsilon$  is given by

$$\epsilon = \sqrt{-2E_n} = \frac{1}{n + \frac{\delta_1 + \delta_2}{2}}, \tag{11}$$

from which the exact energy spectrum

$$E_n \equiv E_n^{(s)} = -\frac{1}{2(n + \frac{\delta_1 + \delta_2}{2})^2} \tag{12}$$

is obtained.

Mardoyan [17, 18] also showed that the Schrödinger equation for the MIC-Kepler system is separated in parabolic coordinates.

### 3 The Schrödinger Ladder Operators and the $su(1, 1)$ Dynamical Algebra

Performing the change  $R_{nj}(r) = \chi_{nj}(r)/r$  on (9), we obtain

$$\left(-r^2 \frac{d^2}{dr^2} - 2r - 2E_n r^2\right) \chi_{nj}(r) = -\left(j + \frac{\delta_1 + \delta_2}{2}\right) \left(j + \frac{\delta_1 + \delta_2}{2} + 1\right) \chi_{nj}(r). \tag{13}$$

We set

$$K_n^2 = -\frac{1}{2E_n} \tag{14}$$

and  $r = K_n x$  to write this equation in the form

$$\mathcal{L}_n \chi_{nj} \equiv \left(-x^2 \frac{d^2}{dx^2} - 2K_n x + x^2\right) \chi_{nj} = -J(J + 1) \chi_{nj}, \tag{15}$$

where  $J \equiv j + \frac{\delta_1 + \delta_2}{2}$ .

In order to factorize the operator  $\mathcal{L}_n$  we apply the Schrödinger factorization [19, 26]. Thus, we propose a pair of first-order differential operators such that

$$\left(x \frac{d}{dx} + ax + b\right) \left(-x \frac{d}{dx} + cx + f\right) \chi_{nj} = g \chi_{nj}, \tag{16}$$

where  $a, b, c, f$  and  $g$  are constants to be determined. Expanding this expression and comparing it with (15) we obtain

$$a = c = \pm 1, \quad b = f - 1 = \mp K_n - 1, \quad g = K_n(K_n \pm 1) - J(J + 1). \tag{17}$$

Using these results (15) is equivalent to

$$(T_-^n - 1)T_+^n \chi_{nj} = [K_n(K_n + 1) - J(J + 1)] \chi_{nj}, \tag{18}$$

$$(T_+^n + 1)T_-^n \chi_{nj} = [K_n(K_n - 1) - J(J + 1)] \chi_{nj}, \tag{19}$$

where we have defined the operators

$$T_{\pm}^n = \mp x \frac{d}{dx} + x - K_n. \tag{20}$$

If we define the operator

$$T_3 = \frac{1}{2} \left( -x \frac{d^2}{dx^2} + x + \frac{J(J+1)}{x} \right) \tag{21}$$

and considering (15) we have

$$T_3 \chi_{nj} = K_n \chi_{nj}. \tag{22}$$

Therefore, this operator allows to define the new operators

$$T_{\pm} = \mp x \frac{d}{dx} + x - T_3. \tag{23}$$

A direct calculation shows that the set of operators  $T_{\pm}$  and  $T_3$  satisfy the commutation relations

$$[T_{\pm}, T_3] = \mp T_{\pm}, \tag{24}$$

$$[T_+, T_-] = -2T_3. \tag{25}$$

Thus, the Schrödinger factorization allowed us to construct the generators of the  $su(1, 1)$  Lie algebra for the generalized MICZ-Kepler problem.

The quadratic Casimir operator

$$T^2 \equiv -T_{\pm}T_{\mp} + T_3^2 \mp T_3, \tag{26}$$

satisfies the eigenvalue equation

$$T^2 \chi_{nj} = J(J+1) \chi_{nj}. \tag{27}$$

In order to obtain the energy spectrum, we consider the theory of unitary irreducible representations of the  $su(1, 1)$  Lie algebra, which has been studied in several works [1] and it is based on the eigenvalue equations

$$C^2 \Psi_{v\mu} = \mu(\mu+1) \Psi_{v\mu}, \tag{28}$$

$$C_3 \Psi_{v\mu} = \nu \Psi_{v\mu}, \tag{29}$$

where  $C^2$  is the quadratic Casimir operator,  $\nu = \mu + n' + 1$ ,  $n' = 0, 1, 2, \dots$  and  $\mu > -1$ .

From (27) and (28) we find  $\mu = J$ . Thus

$$\nu = J + n' + 1 = j + \frac{\delta_1 + \delta_2}{2} + n' + 1. \tag{30}$$

If we define  $n \equiv j + n' + 1$  and by using (14), (22), (29) and (30) we find that the energy spectrum for the MICZ-Kepler problem is

$$E_n = -\frac{1}{2(n + \frac{\delta_1 + \delta_2}{2})^2}. \tag{31}$$

From (8),  $j$  takes integer or half-integer values, hence  $n$  takes integer or half-integer values, which is in accordance with the result obtained by using the confluent hypergeometric function, (12).

Equations (22) and (24) allow to obtain

$$T_3 T_{\pm} \chi_{nj} = (K_n \pm 1) T_{\pm} \chi_{nj}. \quad (32)$$

From (22), (29) and (30), and the definition of  $n$ , we obtain that  $K_n$  satisfies the relation

$$K_{n\pm 1} = K_n \pm 1. \quad (33)$$

Thus, using relation (33), we show

$$T_{\pm} \chi_{nj} \propto \chi_{n\pm 1 j}. \quad (34)$$

From this expression and (22) and (23), we find that the Schrödinger operators,  $T_{\pm}^n$ , acting on the eigenstates  $\chi_{nj}$  satisfy

$$T_{\pm}^n \chi_{nj} \propto \chi_{n\pm 1 j}. \quad (35)$$

In this way we showed the essence of the Schrödinger factorization which is to find operators that relate eigenstates belonging to the same secondary quantum number  $j$  but different principal quantum number  $n$ . It must be pointed out that these results are achieved without the knowledge of the explicit form of the eigenfunctions.

#### 4 Concluding Remarks

We have applied the Schrödinger factorization to the radial equation of the generalized MICZ-Kepler problem and a pair of first-order differential operators were obtained. From the radial Hamiltonian we introduced, in a natural way, a third operator which closes the  $su(1, 1)$  dynamical algebra for this problem. Moreover, we obtained from a purely algebraic way the energy spectrum and showed that the action of the Schrödinger operators on the radial eigenstates is to change only the principal quantum number  $n$  leaving fixed the total angular momentum quantum number  $j$ . Notice that for the construction of these generators we did not introduced any additional variable, which differs from the procedure follow in [10, 16, 21, 29]. To our knowledge the results we have shown in this paper have not been found before.

If we restrict our results to the standard MICZ-Kepler problem, then the  $O(4)$  symmetry of this problem accounts for the degenerate states for a fixed energy, whereas the  $su(1, 1)$  dynamical algebra accounts for the degenerate states for fixed total angular momentum. As it is well known the hydrogen atom possesses the  $su(1, 1)$  dynamical symmetry [30]. Therefore, the existence of the  $su(1, 1)$  dynamical algebra and a Runge-Lenz type vector as constant of motion for the MICZ-Kepler problem makes this system completely similar to the non-relativistic hydrogen atom, at least at algebraic level.

As another application of our method we have studied the relativistic hydrogen atom for which we have constructed the three generators for the  $su(1, 1)$  [25].

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